Pricing and hedging of climate transition risk in Credit Portfolio

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Introduction I: Context

- It is now clear that climate change has and will have adverse effects on the environment, on global warming, and on human societies.
- Different meetings (the Kyoto Protocol in 1997, the Copenhagen Agreement in 2009, the Paris Climate Agreement in 2015, etc.) have progressively allowed us to take the measure of the challenge and to initiate a transition to a low-carbon economy.
- Climate risk has two components:
  - *physical risk*, which is due to changes in climate variables (such as rising temperatures and melting ice) or the occurrence of extreme weather events (such as droughts or typhoons) that can damage infrastructure and endanger populations;
  - *transition risk* which comes from the necessity of transitioning to a low-carbon economy, leading to the implementation of regulatory measures, potential technological changes, and the evolution of consumer preferences.
- Previous financial crises have shown the existence of a strong relationship between the financial sphere and the real one.
- **Transition risk** can modify the three main components of **credit risk**, which are *the borrower's cash flows, the value of his assets, and the value of his guarantees.*

Introduction II: Questions

The aim of this work is to build mathematical models to quantify and project risk measures of a credit portfolio when the bank operate in a economy that undergoes the climate transition.

1. **How to model the climate transition?** The financial sector generally considers three main types of drivers for the transition risk: changes in consumer preferences, technological changes, and political changes.
2. **How to model the default of a borrower and link it to the climate transition?** A bank can trigger the *default* of a firm if the latter faces a *liquidity crisis* or a *insolvency crisis*.
3. **How to model the loss in case of default and link it to the climate transition?** When the bank activates the default of a borrower, the former can lose a part of the loan. This loss is the difference between the amount of exposure at the time of *default* and the liquidation value of the company and its guarantees.

Introduction III: Solution steps

1. **Environment:** The bank as well as the firms in its portfolio and their guarantees, operate in the same economy (e.g. France).
2. **Portfolio segmentation:** We divide our portfolio into I disjoint sub-portfolios $g_1, \ldots, g_I$ so that each sub-portfolio represents a single risk class and that the companies in each of them belong to a single carbon emission sector.
3. **Describing the climate transition:** The carbon intensities τ, κ, ζ and the carbon price δ which are deterministic processes.
4. **Describing the economy:** the economy is assumed to be closed, organized in I sectors, driven by the productivity Θ following a multidimensional Ornstein-Uhlenbeck, and subject to a climate transition modeled through the carbon price δ.
5. **Modelling corporate default:** a given levered firm defaults if it is *over-indebted* i.e. if the market value of debt exceeds the firm's market value (which undergoes the climate transition).
6. **Modelling the bank's loss:** When a company defaults, the bank loses a fraction of the difference between its exposure and the eventual amount obtained by liquidating the collateral (which also undergoes the climate transition).

Plan

A Multisectoral Model with Carbon emissions costs

Introduction

- Time is continuous and economy is closed and divided in $I \in \mathbb{N}^*$ sectors.
- For each sector,
  - firms emit GHG when produce and consume goods or/and services;
  - households emit GHG when consumes goods or/and services.
- Firms as well as households GHG emissions are charged.
- The representative firm in each sector maximizes its profits.
- The representative household maximizes its utility (which is logarithmic in consumption and isoelastic in labour).
- Goods and labour markets clear.

The main goal is to derive the dynamics of macroeconomic variables (greenhouse gases emissions, output, consumption, labour, and intermediary inputs) per sector and per carbon price scenario.

The productivity process

Assumption

We define the $\mathbb{R}^I$*-valued process* $\mathcal{A}$ *which evolves according to*

$$\begin{cases} \mathrm{d}\mathcal{Z}_t &= -\Gamma \mathcal{Z}_t \mathrm{d}t + \Sigma \mathrm{d}B_t^{\mathcal{A}} \\ \mathrm{d}\mathcal{A}_t &= (\mu + \varsigma \mathcal{Z}_t)\mathrm{d}t \end{cases} \quad \textit{for all } t \in \mathbb{R}^*, \tag{1}$$

where

- $(B_t^{\mathcal{A}})_{t \in \mathbb{R}^*}$ *is a I-dimensional Brownian Motion,*
- *the constants* $\mu, \mathcal{A}_0 \in \mathbb{R}^I$*,*
- $\mathcal{Z}_0 \sim \mathcal{N}\left(0, \Sigma\Sigma^\top\right)$
- $0 < \varsigma \leq 1$ *is an intensity of noise parameter,*
- $\Sigma \in \mathbb{R}^{I \times I}$ *is a positive definite matrix,*
- $-\Gamma \in \mathbb{R}^{I \times I}$ *is a Hurwitz matrix i.e. its eigenvalues have negative real parts.*

We also introduce the following filtration $\mathbb{G} := (\mathcal{G}_t)_{t \in \mathbb{N}}$ with $\mathcal{G}_0 := \sigma(\mathcal{Z}_0)$ and for $t > 0$, $\mathcal{G}_t := \sigma\left(\left\{\mathcal{Z}_0, B_s^{\mathcal{A}} : s \leq t\right\}\right)$.

The transition scenario

Assumption

1. **Carbon prices δ:** *let $0 \leq t_\circ < t_\star$ be given. The sequence δ satisfies*
   - *for $t \in [0; t_\circ]$, $\delta_t = \delta_0 \in \mathbb{R}_+$, namely the carbon price is constant;*
   - *for $t \in (t_\circ, t_\star)$, $\delta_t \in \mathbb{R}_+$, the carbon price may evolve;*
   - *for $t \geq t_\star$, $\delta_t = \delta_{t_\star} \in \mathbb{R}_+$, namely the carbon price is constant.*

   We assume moreover that $t \mapsto \delta_t$ is $\mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$.
2. **Carbon intensity:**
   - *of firm production $\tau \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+^I)$*
   - *of firm intermediate consumption $\zeta \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+^{I\times I})$*
   - *of household consumption $\kappa \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+^I)$*

   satisfying for all $\mathfrak{y} \in \{\tau^1, \ldots, \tau^I, \zeta^{11}, \zeta^{12}, \ldots, \zeta^{II-1}, \zeta^{II}, \kappa^1, \ldots, \kappa^I\}$,

$$\mathfrak{y}_t = \begin{cases} \mathfrak{y}_0 \exp\left\{\left(g_{\mathfrak{y},0} \frac{1-\exp\{(-\theta_{\mathfrak{y}} t)\}}{\theta_{\mathfrak{y}}}\right)\right\} & \text{if } t \leq t_\star \\ \mathfrak{y}_0 \exp\left\{\left(g_{\mathfrak{y},0} \frac{1-\exp\{(-\theta_{\mathfrak{y}} t_\star)\}}{\theta_{\mathfrak{y}}}\right)\right\} & \text{else,} \end{cases} \tag{2}$$

   with $\mathfrak{y}_0, -g_{\mathfrak{y},0}, \theta_{\mathfrak{y}} > 0$.

In the following, we will note for all $t \geq 0$, $\mathfrak{d}_t := (\tau_t\delta_t, \zeta_t\delta_t, \kappa_t\delta_t)$. Therefore, the process $\mathfrak{d}$ is $\mathcal{C}^1(\mathbb{R}_+, [0,1)^I \times (\mathbb{R}_+)^{I\times I} \times (\mathbb{R}_+)^I)$.

Theorem

For $(\overline{\tau}, \overline{\zeta}, \overline{\kappa}, \overline{\delta}) \in \mathbb{R}_+^I \times \mathbb{R}_+^{I\times I} \times \mathbb{R}_+^I \times \mathbb{R}_+$*, let us pose* $\overline{\mathfrak{d}} := (\overline{\tau\delta}, \overline{\zeta\delta}, \overline{\kappa\delta})$*. We also pose*

$$\Psi(\overline{\mathfrak{d}}) := \left(\psi^i \frac{1-\overline{\tau}^i\overline{\delta}}{1+\overline{\kappa}^i\overline{\delta}}\right)_{i\in\mathcal{I}} \qquad \Lambda(\overline{\mathfrak{d}}) := \left(\boldsymbol{\lambda}^{ji} \frac{1-\overline{\tau}^i\overline{\delta}}{1+\overline{\zeta}^{ji}\overline{\delta}} \frac{1+\overline{\kappa}^j\overline{\delta}}{1+\overline{\kappa}^i\overline{\delta}}\right)_{j,i\in\mathcal{I}}, \tag{3}$$

Assume that:

1. $\sigma = 1$,
2. $\mathbf{I}_I - \boldsymbol{\lambda}$ *is not singular,*
3. $\mathbf{I}_I - \Lambda(\mathfrak{d}_t)^\top$ *is not singular for all* $t \geq 0$.

Then, for all $t \geq 0$,

- *the consumption is* $C_t = \exp\left\{\left((\mathbf{I}_I - \boldsymbol{\lambda})^{-1}\mathcal{A}_t + v(\mathfrak{d}_t)\right)\right\}$,
- *the output is* $Y_t = \mathrm{Diag}(\mathfrak{e}_t)C_t$,
- *the labor is* $N_t = [\mathrm{Diag}(\Psi(\mathfrak{d}_t))\mathfrak{e}_t]^{\frac{1}{1+\varphi}}$,
- *and the intermediary inputs* $Z_t = \Lambda(\mathfrak{d}_t) \odot C_t\mathfrak{e}_t^\top$,

where

$$\mathfrak{e}(\overline{\mathfrak{d}}) := (\mathbf{I}_I - \Lambda(\overline{\mathfrak{d}})^\top)^{-1}\mathbf{1} \quad \text{and} \quad v(\overline{\mathfrak{d}}) = \left[\log\left(\left(\mathfrak{e}(\overline{\mathfrak{d}})^i\right)^{-\frac{\phi\psi^i}{1+\phi}} \left(\Psi^i(\overline{\mathfrak{d}})\right)^{\frac{\psi^i}{1+\phi}} \prod_{j\in\mathcal{I}} \left(\Lambda^{ji}(\overline{\mathfrak{d}})\right)^{\boldsymbol{\lambda}^{ji}}\right)\right]_{i\in\mathcal{I}}. \tag{4}$$

Loss with stochastic collateral in continuous time

- **Different types of default :** a levered firm is
  - *in danger of illiquidity* if the cash flows do not suffice to fulfill the creditors' payment claims,
  - *over-indebted* if the market value of debt exceeds the firm's market value.
- **Different types of loans:**
  - *unsecured loans:* there is not collateral, bank tries to recover Exposure at default (EAD) through different processes.
  - *secured loans:* when the counterpart defaults, the bank liquidates the collateral (guarantee), and if the EAD is not reached, it can recover the remainder by liquidating other assets (residual recovery).
- **Different types of collateral:**
  - *tangible assets:* buildings, business equipments, inventories, etc.
  - *intangible assets:* cash deposits, public bonds, securities, etc.
- **Collateral and transition risk:**
  - a commercial or residential building price will be affected by transition for example through the Energy Performance (or Energy Efficiency).

Loss with collateral

- We consider a portfolio composed by $N \in \mathbb{N}^*$ loans indexed by n.
- For each n and for each date $t \geq 0$:
  - the loan n corresponds to a firm n;
  - $\mathcal{D}_t^n$ is the debt of n at t
  - V_t^n is the market value of n at t;
  - EAD_t^n the exposure of the bank to n at t.
- $k \in [0, 1)$ represents costs due to liquidations auctions, as well as other legal and administrative costs.
- r is the discount rate; $\gamma \in [0, 1)$ the residual recovery; $a \geq 0$ is the liquidation period.

If firm n defaults at t, i.e. $V_t^n < \mathcal{D}_t^n$, the bank recovers $(1-k)e^{-ra}C_{t+a}^n$ by the collateral liquidation and if it is not enough, $\gamma(\mathrm{EAD}_t^n - (1-k)e^{-ra}C_{t+a}^n)_+$ by other ways. The loss due to firm n at time t is

$$L_{n,t} = (1-\gamma)(\mathrm{EAD}_t^n - (1-k)e^{-ra}C_{t+a}^n)_+ \cdot \mathbf{1}_{\{V_t^n < \mathcal{D}_t^n\}} \tag{5}$$

How to compute V_t^n, C_{t+a}^n, then $L_{n,t}$ when n operate in the above economy?

A Firm Valuation Model to compute V_t^n.

Assumption and definition

Consider a fixed $n \in \{1, \ldots, N\}$. For any time $t \in \mathbb{R}_+$ and firm n, we note F_t^n the free cash flows of n at t, and $r > 0$ the discount rate, we introduce the following assumption:

Assumption

The $\mathbb{R}$-valued process on the instantaneous growth of the cash flows of firm n denoted by $\mathrm{d}\log F_{t,\mathfrak{d}}^n$ is linear in the economic factors (output growth by sector), specifically we set for all $t \in \mathbb{R}_+$,

$$\mathrm{d}\log F_{t,\mathfrak{d}}^n = \tilde{\mathfrak{a}}^{n\cdot}\mathrm{d}\log Y_t + \sigma_n \mathrm{d}\mathcal{W}_t^n = \mathfrak{a}^{n\cdot}(\mathrm{d}\mathcal{A}_t + \mathrm{d}v(\mathfrak{d}_t)) + \sigma_n \mathrm{d}\mathcal{W}_t^n, \tag{6}$$

for $\tilde{\mathfrak{a}}^{n\cdot} \in \mathbb{R}^I$ and $\mathfrak{a}^{n\cdot} = \tilde{\mathfrak{a}}^{n\cdot}(\mathbf{I}_I - \boldsymbol{\lambda})^{-1}$, where $(\mathcal{W}_t^n)_{t\in\mathbb{R}_+}$ is a $\mathbb{R}^N$-Brownian motion with $\sigma_n > 0$. Moreover, $B^{\mathcal{A}}$ and $\mathcal{W}^n$ are independent.

We define the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ by $\mathcal{F}_t = \sigma\left(\mathcal{G}_t \cup \sigma\left\{\mathcal{W}_s^n : s \in [0,t]\right\}\right)$ for $t \geq 0$, and we denote $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$.

Definition

Let $r \geq 0$ representing the interest rate, by the continuous form of the discounted cash flows valuation, the value $V_{t,\mathfrak{d}}^n$ of the firm n, at time t, is

$$\begin{aligned} V_{t,\mathfrak{d}}^n &:= \mathbb{E}_t\left[\int_t^{+\infty} e^{-r(s-t)} F_{s,\mathfrak{d}}^n \mathrm{d}s\right] \\ &= F_{t,\mathfrak{d}}^n \int_t^{+\infty} e^{-r(s-t)} \mathbb{E}_t\left[\exp\left(\mathfrak{a}^{n\cdot}(\mathcal{A}_s - \mathcal{A}_t) + \mathfrak{a}^{n\cdot}\left(v(\mathfrak{d}_s) - v(\mathfrak{d}_t)\right) + \sigma_n(\mathcal{W}_s^n - \mathcal{W}_t^n)\right)\mathrm{d}s\right]. \end{aligned}$$

Definition

How to compute $V^n_{t,\mathfrak{d}}$? Picard iteration, Deep learning, or an approximation method.
For $s \geq t \geq 0$, inspired by (**??**) i.e. $\mathcal{A}_s - \mathcal{A}_t = (s-t)\mu + \varsigma \int_t^s \mathcal{Z}_u \mathrm{d}u$, we can introduce the quantity

$$\mathcal{V}^n_{t,\mathfrak{d}} := F^n_{t,\mathfrak{d}} \int_t^{+\infty} e^{-r(s-t)} \mathbb{E}_t \left[\exp\left((s-t)\mathfrak{a}^{n \cdot} \mu + \mathfrak{a}^{n \cdot} \left(v(\mathfrak{d}_s) - v(\mathfrak{d}_t) \right) + \sigma_n (\mathcal{W}^n_s - \mathcal{W}^n_t) \right) \mathrm{d}s \right]. \tag{7}$$

Proposition

For any $n \in \{1, \dots, N\}$ and for all $t \in \mathbb{R}_+$.

1. *Assume that $\varrho_n := \frac{1}{2}\sigma_n^2 + \mathfrak{a}^{n \cdot}\mu - r < 0$, then $\mathcal{V}^n_{t,\mathfrak{d}}$ is well defined and*

$$\mathcal{V}^n_{t,\mathfrak{d}} = F^n_0 \mathfrak{R}^n_t(\mathfrak{d}) \exp\{ (\mathfrak{a}^{n \cdot} (\mathcal{A}_t - v(\mathfrak{d}_0))) \} \exp\left(\sigma_n \mathcal{W}^n_t \right), \tag{8}$$

   where

$$\mathfrak{R}^n_t(\mathfrak{d}) := \int_0^\infty e^{\varrho_n s} \exp\{ (\mathfrak{a}^{n \cdot} v(\mathfrak{d}_{t+s})) \} \mathrm{d}s. \tag{9}$$

2. *Assume that $\rho_n := \frac{1}{2}\sigma_n^2 + \mathfrak{a}^{n \cdot}\mu + \frac{1}{2}\varsigma^2 \frac{c_\Gamma^2}{\lambda_\Gamma^2} \|\mathfrak{a}^{n \cdot}\|^2 \|\Sigma\|^2 < r$, therefore $V^n_{t,\mathfrak{d}}$ is well defined and there exists a constant C such that $\mathbb{E}\left[\left| \frac{V^n_{t,\mathfrak{d}}}{F^n_{t,\mathfrak{d}}} - \frac{\mathcal{V}^n_{t,\mathfrak{d}}}{F^n_{t,\mathfrak{d}}} \right| \right] \leq C\varsigma$, for all $\varsigma > 0$.*

A real estate model to compute C_t^n.

Assumptions

1. in the absence of climate transition, the housing price follows an Exponential Ornstein-Uhlenbeck process;
2. each dwelling consumes a given quantity of energy per square meter, which is used to determine its energy efficiency, noted α and expressed in kilowatts per square meter (KWh/m^2);
3. as a consequence, the dwelling price is depreciated (or appreciated) by the actualized sum of the future energy costs;
4. once a certain level of energy efficiency $\alpha^\star$ is reached, the market is insensitive to this factor;
5. the energy price is a deterministic function $\mathfrak{f}$ of two variables, the first variable is the carbon price and the second is the source of energy,
6. during the life of the property, the owner may undergo renovations which moves the energy efficiency from α to $\alpha^\star$, and whose cost per square meter is a function $\mathfrak{c}$ of its energy efficiency,
7. the date of renovation $\mathfrak{t}$ of a dwelling is unknown, but is to be optimized.
8. after renovations, the price of the building becomes insensitive to energy costs.

The model: the sales comparison approach

This consists in using the transaction prices of highly comparable and recently sold properties to determine the market value of a given building.

Assumption (Housing price without climate transition)

The market value of the building at $t \geq 0$, is given by

$$C_t := RC_0 e^{K_t}, \tag{10}$$

where

$$\mathrm{d}K_t = \left(\dot{\chi}_t + \nu(\chi_t - K_t)\right)\mathrm{d}t + \overline{\sigma}\mathrm{d}\overline{B}_t, \tag{11a}$$

$$\mathrm{d}\overline{B}_t = \rho^\top \mathrm{d}B^{\mathcal{A}}_t + \sqrt{1 - \|\rho\|^2}\mathrm{d}\overline{\mathcal{W}}_t, \tag{11b}$$

where $(\overline{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion independent to $B^{\mathcal{A}}$. Moreover, C_0, r, $R, \overline{\sigma} > 0$, $\rho \in \mathbb{R}^I_+$, and $\chi \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$.

We introduce the following filtration $\mathbb{U} := (\mathcal{U}_t)_{t\in\mathbb{R}^*}$ with for $t \geq 0$,
$\mathcal{U}_t := \sigma\left(\left\{\overline{W}_s, B^{\mathcal{A}}_s : s \leq t\right\}\right)$.

The *income approach* with climate transition

In this case, the value of a property is the discounted sum of the (imputed) rent less all operating costs over its residual lifetime.

Assumption (Housing price without climate transition)

Owning a building allows to generate an (imputed) income cash-flow process $(D_t)_{t \geq 0}$ which is continuous and $\mathbb{U}$-adapted, therefore for all $t \geq 0$, an another way to write the price C_t of the building is

$$C_t = R\mathbb{E}\left[\int_0^{+\infty} e^{-\bar{r}s} D_{t+s} \mathrm{d}s \middle| \mathcal{U}_t\right], \tag{12}$$

with $\bar{r} > 0$.

Definition (Housing price with climate transition)

The market value of the building at $t \geq 0$, given the carbon price sequence δ, is represented by

$$\mathcal{C}_{t,\delta} := R \operatorname*{ess\,sup}_{\theta \geq t} \mathbb{E}\left[\begin{array}{c} \int_t^\theta \left[D_s - (\alpha - \alpha^\star)\mathfrak{f}(\delta_u, \mathfrak{p})\right] e^{-\bar{r}(u-t)} \mathrm{d}u - \mathfrak{c}(\alpha, \alpha^\star) e^{-\bar{r}(\theta - t)} \\ + \int_\theta^{+\infty} e^{-\bar{r}(s-t)} D_{t+s} \mathrm{d}s \end{array} \middle| \mathcal{U}_t\right], \tag{13}$$

where

- $\mathfrak{c}$ and $\mathfrak{f}(., \mathfrak{p})$ (for each source of energy $\mathfrak{p}$) are regular,
- $r, \alpha, \alpha^\star > 0$ with $\alpha > \alpha^\star$.

Main result: the housing price under climate transition

Theorem

Assume that the following conditions are satisfied:

1. *the carbon price function* $\delta : t \mapsto \delta_t$ *is non decreasing on* $\mathbb{R}_+$ *and deterministic;*
2. *the energy price* $\mathfrak{f}(\cdot, \mathfrak{p})$ *is non decreasing on* $\mathbb{R}_+$ *for all* $\mathfrak{p}$*.*

Then, the market value of the building serving as the collateral to firm n *at* $t \geq 0$*, given the carbon price sequence* δ*, is given by*

$$\mathcal{C}_{t,\delta} = C_t - RX_{t,\delta}, \tag{14}$$

where

$$X_{t,\delta} := \mathfrak{c}(\alpha, \alpha^\star) e^{-\bar{r}(\theta^\star - t)} + (\alpha - \alpha^\star) \int_t^{\theta^\star} \mathfrak{f}(\delta_u, \mathfrak{p}) e^{-\bar{r}(u-t)} \mathrm{d}u, \tag{15}$$

and where the optimal date of renovations $\mathfrak{t} \in [t, +\infty]$ *is given by*

$$\theta^\star = \begin{cases} t & \text{if } \mathfrak{f}(\delta_\theta, \mathfrak{p}) - \bar{r}\frac{\mathfrak{c}(\alpha,\alpha^\star)}{\alpha - \alpha^\star} > 0 \text{ for all } \theta \in [t, \infty) & (16) \\ +\infty & \text{if } \mathfrak{f}(\delta_\theta, \mathfrak{p}) - \bar{r}\frac{\mathfrak{c}(\alpha,\alpha^\star)}{\alpha - \alpha^\star} < 0 \text{ for all } \theta \in [t, \infty) & (17) \\ \theta^\star & \text{the unique solution of } \mathfrak{f}(\delta_\theta, \mathfrak{p}) = \bar{r}\frac{\mathfrak{c}(\alpha,\alpha^\star)}{\alpha - \alpha^\star} \text{ on } \theta \in [t, \infty) & (18) \end{cases}$$

A credit loss model to compute L_t^n.

Definition of PD and LGD

The loss of the portfolio at time t, is in fact, defined as

$$L_t^N := \sum_{n=1}^{N} L_{n,t} = \sum_{n=1}^{N} (1-\gamma)(\mathrm{EAD}_t^n - (1-k)e^{-ra}\mathcal{C}_{t+a}^n)_+ \cdot \mathbf{1}_{\{\mathcal{V}_{t,\mathfrak{d}}^n < \mathcal{D}_t^n\}}. \tag{19}$$

For all $t \in \mathbb{R}_+$, define

$$\mathrm{L}_t^{\mathbb{G},N} := \sum_{n=1}^{N} L_{n,t}^{\mathbb{G}} \quad \text{with} \quad L_{n,t}^{\mathbb{G}} = \mathbb{E}\left[L_{n,t} | \mathcal{G}_t\right] = \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \mathrm{PD}_t^n, \tag{20}$$

where

$$\mathrm{PD}_t^n := \mathbb{P}\left(\mathcal{V}_{t,\mathfrak{d}}^n < \mathcal{D}_t^n | \mathcal{G}_t\right), \tag{21a}$$

$$\mathrm{LGD}_t^n := (1-\gamma)\mathbb{E}\left[\left(1-(1-k)e^{-ra}\frac{C_{t+a}^n}{\mathrm{EAD}_t^n}\right)_+ \middle| \mathcal{V}_{t,\mathfrak{d}}^n < \mathcal{D}_t^n, \mathcal{G}_t\right]. \tag{21b}$$

Probability of Default

Proposition

For each firm n, assume that

- *Firm n has issued two classes of securities: equity and debt;*
- $(\mathrm{EAD}_t^n)_{t\in\mathbb{N}^*}$ *is a* $\mathbb{R}_*^+$*-valued deterministic process representing the bank's exposure to the firm n;*
- *the default barrier* $\mathcal{D}^n \in \mathbb{R}^+$ *is a deterministic scalar;*
- *the value of the firm n at time t is assumed to be a tradable asset given by* $\mathcal{V}_t^n$.

Assume also that, for all $t \geq 0$,

1. $\sum_{n\geq 1} \mathrm{EAD}_t^n = +\infty$;
2. *there exists* $\upsilon > 0$ *such that* $\frac{\mathrm{EAD}_t^n}{\sum_{n=1}^N \mathrm{EAD}_t^n} = \mathcal{O}(N^{-(\frac{1}{2}+\upsilon)})$, *as N tends to infinity.*

Therefore, $\mathrm{L}_t^N - \mathrm{L}_t^{\mathbb{G},N}$ *converges to zero almost surely as N tends to infinity.*

Proposition

For each firm n and at each date t, the conditional (on the productivity and the carbon price) probability that n defaults at t is

$$\mathrm{PD}_t^n := \Phi\left(\frac{\log(\mathcal{D}^n) - \mathfrak{m}^n(\mathfrak{d}, t, \mathcal{A}_t^\circ)}{\sigma_n\sqrt{t}}\right), \tag{22}$$

where $\mathfrak{m}^n(\mathfrak{d}, t, \mathcal{A}_t^\circ) := \mathbb{E}[\log \mathcal{V}_{t,\mathfrak{d}}^n | \mathcal{G}_t]$.

Loss Given Default for real estate collateral

- The collateral of the firm n is a building.
- The latter can be considered as an asset in the economy described above.
- Its surface is R_n, its price at time 0 is C_0^n, and its energy efficiency is α^n.
- We can determine the collateral value by Theorem **??**.

Theorem

Let $1 \leq n \leq N$. When $a = 0$ (no liquidation delay), the Loss Given Default of the obligor n is over-indebted at time $t \in \mathbb{R}_+$, conditional on $\mathcal{G}_t$, is

$$\mathrm{LGD}_{t,\delta}^n = (1-\gamma)\left[\left(1 + (1-k)\frac{R_n X_{t,\delta}^n}{\mathrm{EAD}_t^n}\right)\Phi\left(\frac{w_t^n}{\sqrt{v_{t,0}^n}}\right) - \exp\left\{\left(-w_t^n + \frac{1}{2}v_{t,0}^n\right)\right\}\Phi\left(\frac{w_t^n}{\sqrt{v_{t,0}^n}} - \sqrt{v_{t,0}^n}\right)\right], \tag{23}$$

where

$$w_t^n := \log\left(\frac{\mathrm{EAD}_t^n}{(1-k)R_n} + X_{t,\delta}^n\right) - m_{t,0}^n, \tag{24}$$

*and with $m_{t,0}^n := \log\left(R_n C_0^n\right) + \mathbb{E}[K_t|\mathcal{G}_t]$ and $v_{t,t}^n := \mathbb{V}[K_t|\mathcal{G}_t]$, and $X_{t,\delta}^n$ defined in (**??**).*

Expected and unexpected losses

Let $t \geq 0$ be the time when the risk measure is computed for a period $T \geq 0$. As classically done, the potential loss is separated into three components:

1. The (conditional) Expected Loss (EL) reads

$$\mathrm{EL}_t^{N,T} := \mathbb{E}\left[\mathrm{L}_{t+T}^{\mathbb{G},N} - \mathrm{L}_t^{\mathbb{G},N} \middle| \mathcal{G}_t\right]. \tag{25}$$

2. The Unexpected Loss (UL) reads for $\alpha \in (0,1)$,

$$\mathrm{UL}_t^{N,T}(\alpha) := \mathrm{VaR}_t^{\alpha,N,T} - \mathrm{EL}_t^{N,T}, \quad \text{where} \quad 1-\alpha = \mathbb{P}\left[L_{t+T}^{\mathbb{G},N} - \mathrm{L}_t^{\mathbb{G},N} \leq \mathrm{VaR}_t^{\alpha,N,T} \middle| \mathcal{G}_t\right]. \tag{26}$$

3. The Stressed Loss (or Expected Shortfall or ES) reads $\mathrm{VaR}_t^{\alpha}(L_s^N)$:

$$\mathrm{ES}_t^{N,T}(\alpha) := \mathbb{E}\left[L_{t+T}^{\mathbb{G},N} - \mathrm{L}_t^{\mathbb{G},N} \middle| L_{t+T}^{\mathbb{G},N} - \mathrm{L}_t^{\mathbb{G},N} \geq \mathrm{VaR}_t^{\alpha,N,T}, \mathcal{G}_t\right], \qquad \text{for } \alpha \in (0,1). \tag{27}$$

Application on fake portfolio

Sectors

The 38 INSEE sectors are grouped into four categories

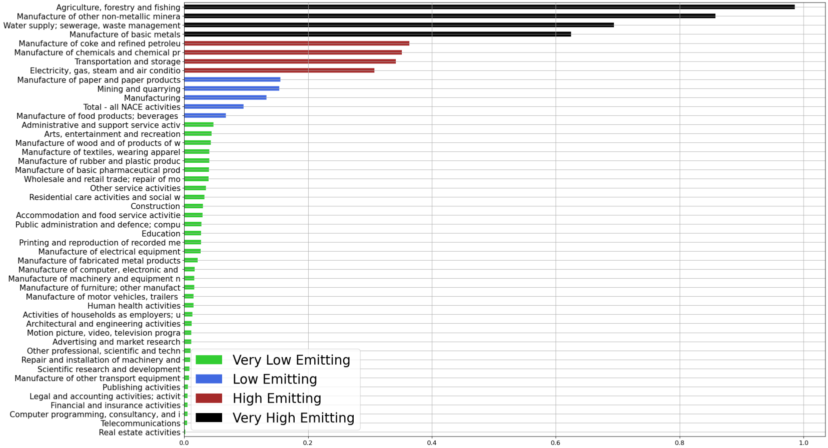


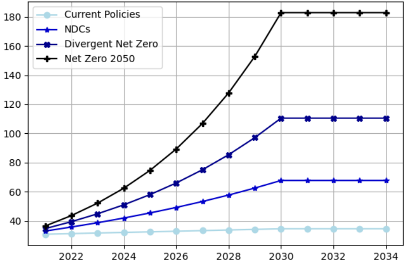
Figure 1: Average air emissions intensities by sectors

Transition scenarios

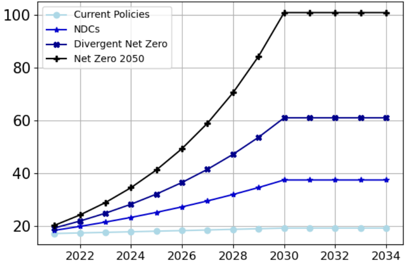
- **An example of carbon price process** We assume the regulator fixes $t_\circ \geq 0$ $t_\star > t_\circ$, $P_{carbon} > 0$, and $\eta_\delta > 0$. Then, for all $t \geq 0$,

$$\delta_t = \begin{cases} P_{carbon}, & \text{if } t \leq t_\circ, \\ P_{carbon} e^{\eta_\delta (t - t_\circ)}, & \text{if } t \in (t_\circ, t_\star], \\ \delta_{t_\star} = P_{carbon} e^{\eta_\delta (t_\star - t_\circ)}, & \text{otherwise.} \end{cases} \tag{28}$$

- The 4 scenarios used come from the NGFS simulations: **Net Zero 2050, Divergent Net Zero, Nationally Determined Contributions (NDCs), Current Policies**.



(a) Carbon price in Euros per tonne



(b) Energy price in Euros per Kilowatt-hour

Figure 2: Per scenario and par year

Description of the portfolio

Firm	1	2	3	4
σ_n	0.05	0.05	0.05	0.05
F_0^n	1.0	1.0	1.0	1.0
α^n**(Very High)**	1.0	0.0	0.0	0.0
α^n**(High)**	0.0	1.0	0.0	0.0
α^n**(Low)**	0.0	0.0	1.0	0.0
α^n**(Very Low)**	0.0	0.0	0.0	1.0

Table 1: Characteristics of the portfolio

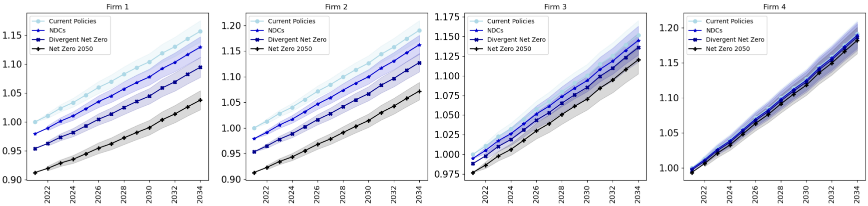


Figure 3: Firm value per scenario and per year

Emissions level	Very High	High	Low	Very Low	Portfolio
NDCs	0.451	0.305	0.096	0.021	0.218
Net Zero 2050	1.419	0.925	0.280	0.062	0.671
Divergent Net Zero	2.625	1.375	0.490	0.120	1.152

Table 2: Average annual PD

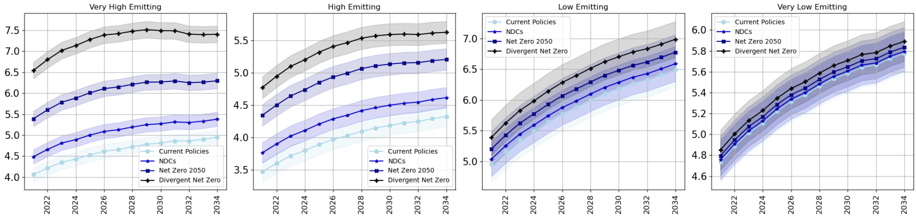


Figure 4: Average annual probability of *default* and 95% confident interval, per scenario and per sub-portfolio

Building's value under climate transition

Building	1	2	3	4	5
C_n^0	4000	4000	4000	4000	4000
α^n	320.	253.	187.	120.	70.
R_n	25.0	25.0	25.0	25.0	25.

Table 3: Characteristics of the building

- **An example of the energy price function:** the price of each type of energy $\mathfrak{p}$ is linear with the carbon price, therefore $\mathfrak{f}(\delta_t, \mathfrak{p}) = \mathfrak{f}_1^{\mathfrak{p}} \delta_t + \mathfrak{f}_0^{\mathfrak{p}}$ for all $t \geq 0$ for $\mathfrak{f}_1^{\mathfrak{p}}, \mathfrak{f}_0^{\mathfrak{p}} > 0$.
- **An example of the renovation costs function:** to move its energy efficiency from α to $\alpha^\star$, the owner pays $\mathfrak{c}(\alpha, \alpha^\star) = c_0|\alpha - \alpha^\star|^{1+c_1}$, with $c_0 > 0$ and $c_1 \geq -1$.

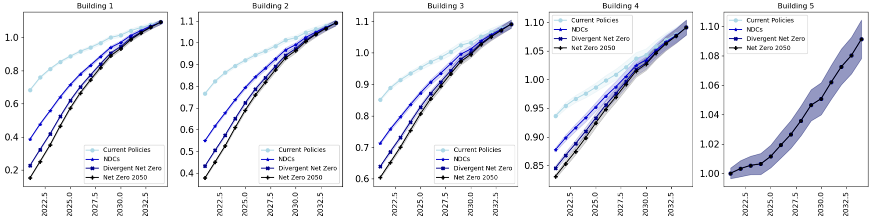


Figure 5: Apartment value per scenario and per year

LGD when collateral is a building

Emissions level	No collateral	Building 1	Building 2	Building 3	Building 4
Current Policies	45.	36.020	36.152	35.928	36.095
NDCs	45.	38.383	37.752	36.922	36.499
Divergent Net Zero	45.	38.939	38.303	37.377	36.751
Net Zero 2050	45.	39.102	38.524	37.615	36.908

Table 4: verage annual LGD per scenario between 2021 and 2030 (in %)

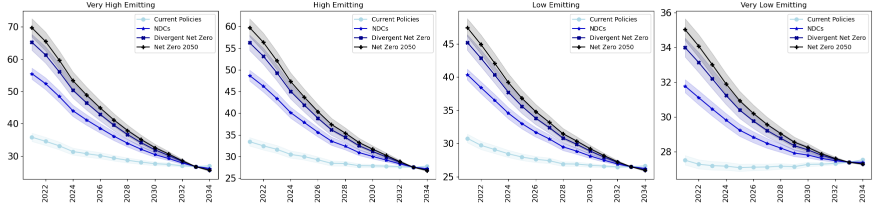


Figure 6: LGD with a building as collateral

Conclusion

- This work provides an end-by-end methodology to calculate the evolution of risk measures of a multisectoral credit portfolio, starting from a given climate transition scenario described by the carbon price.
- We considered that time is continuous and some loans are secured by collateral living to the same economy as borrowers and experiencing the same shocks (productivity and carbon price). We then apply the methodology to the case where the collateral is a building.
- Various tools were used, including macroeconomic modeling, credit portfolio modeling, stochastic optimization, numerical simulations, and risk measures.
- Implementations were carried out in Python on both real and fake data.
- There are many possible extensions: endogenous and stochastic carbon prices, capital as a factor of production, companies optimizing their GHG emission levels, housing being renovated in several steps, EAD also undergoing the transition, physical risk, etc.

Thank you for your attention.

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